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Properties of two-mode nonlinear squeezed vacuum and coherent squeezed vacuum states

Tong-qiang Song[1] **and Hongyi Fan**[2,3]

[1] Department of Physics, Ningbo University, Ningbo, Zhejiang, People's Republic of China
[2] CCAST (World Laboratory), PO Box 8730, Bejing 100080, People's Republic of China
[3] Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui, People's Republic of China

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Abstract
The construction of two f-analogues of the two-mode squeezed vacuum and coherent squeezed vacuum states are derived using deformation quantization methods. The statistical properties of these states are studied and the method of integration within an ordered product is used to derive several new completeness relations.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the past few years much attention has been paid to the q-deformed boson oscillator and its possible applications in studying the q-analogue of the quantum theory of the radiation field [1–5]. Some important physical concepts such as the coherent state, even and odd coherent state and squeezed state have been extended to the q-deformed case. The q-oscillators are nonlinear oscillators with a very specific type of nonlinearity, in which the frequency of vibration depends on the energy of these vibrations through the hyperbolic cosine function containing a nonlinearity parameter. This interpretation of q-oscillators becomes obvious if one uses the classical counterpart of the original quantum q-oscillators. This observation suggests that there might exist other types of nonlinearity for which the frequency of oscillation varies with the amplitude via a generic function; this leads to the concept of f-oscillators devised in [6]. Then the notion of f-coherent states was straightforwardly introduced. The so-called nonlinear coherent states, which are right-hand eigenstates of the product of the boson annihilation operator and a nonlinear function f of the number operator, may be regarded as a generalization of the f-coherent states [7]. Recently, de Matos Filho and Vogel [8] have shown that one special class of nonlinear coherent states could be generated as stationary states of the centre-of-mass motion of a laser-driven trapped ion far from the Lamb–Dicke regime.

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The speciality of these nonlinear coherent states are that besides coherence properties, they can exhibit nonclassical features such as amplitude squeezing and self-splitting accompanied by pronounced quantum interference effects. The notion of nonlinear coherent states has been generalized to the even and odd nonlinear coherent states [9–11] and it was found that these states have rather different statistical properties from those of the usual even and odd coherent states.

Squeezed states are characterized by the property that one of the uncertainties is smaller than that in a coherent state (naturally at the expense of the other, because of Heisenberg's principle). In the past few years, squeezed states have attracted considerable attention due to their promising applications in quantum communication and detection of weak signals [12–16]. In this paper, we will generalize the notion of nonlinear coherent states to the two-mode nonlinear squeezed states, which is based on the properties of the inverses of the annihilation operator and the creation operator of f-oscillators, and study their statistical properties. Moreover, using the technique of integration within an ordered product (IWOP) of operators [17] we derive some new completeness relations.

2. Two-mode nonlinear squeezed states

The annihilation operator b_i and the creation operator b_i^+ of f-oscillators for the ith mode are distortions of the annihilation and creation operators a_i and a_i^+ of the usual harmonic oscillator and are given by [9–11]

$$b_i = a_i f(N_i) \qquad b_i^+ = f(N_i) a_i^+ \qquad i = 1, 2 \tag{1}$$

where

$$N_i = a_i^+ a_i \qquad [b_i, N_i] = b_i \qquad [b_i^+, N_i] = -b_i^+ \tag{2}$$

f being an operator-valued function of the number operators (here it is chosen to be real). The commutator between b_i and b_i^+ can be easily obtained using the representations in Fock space

$$b_i = \sum_{n=0}^{\infty} \sqrt{n+1} f(n+1) |n\rangle_{ii}\langle n+1| \tag{3}$$

$$b_i^+ = \sum_{n=0}^{\infty} \sqrt{n+1} f(n+1) |n+1\rangle_{ii}\langle n| \tag{4}$$

and it reads

$$[b_i, b_i^+] = (N_i + 1) f^2(N_i + 1) - N_i f^2(N_i) \qquad i = 1, 2. \tag{5}$$

We now introduce the inverse of the operators b_i and b_i^+ as follows:

$$b_i^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1} f(n+1)} |n+1\rangle_{ii}\langle n| \tag{6}$$

$$(b_i^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1} f(n+1)} |n\rangle_{ii}\langle n+1| = (b_i^{-1})^+. \tag{7}$$

A noncommutative relation between the inverse of the operators b_i and b_i^+ follows

$$b_i b_i^{-1} = (b_i^+)^{-1} b_i^+ = 1 \tag{8}$$

$$b_i^{-1} b_i = b_i^+ (b_i^+)^{-1} = 1 - |0\rangle_{ii}\langle 0| \tag{9}$$

which means that b_i^{-1} is the right inverse of b_i and $(b_i^+)^{-1}$ is the left inverse of b_i^+. This conclusion is analogous to the case of the inverse of ordinary boson operators. Using the inverse operators b_i^{-1}, $(b_i^+)^{-1}$ and the number operators N_i we define the following operators:

$$B_i^+ = N_i b_i^{-1} \qquad B_i = (b_i^+)^{-1} N_i \qquad i = 1, 2. \tag{10}$$

From equations (2), (8) and (9), we have

$$[b_i, B_i^+] = b_i N_i b_i^{-1} - N_i b_i^{-1} b_i = (N_i + 1) b_i b_i^{-1} - N_i (1 - |0\rangle_{ii}\langle 0|) = 1 \tag{11}$$

$$[B_i, b_i^+] = (b_i^+)^{-1} N_i b_i^+ - b_i^+ (b_i^+)^{-1} N_i = (b_i^+)^{-1} b_i^+ (N_i + 1) - (1 - |0\rangle_{ii}\langle 0|) N_i = 1. \tag{12}$$

Following essentially the same method as in [18], we can prove that the normal product form of the two-mode vacuum projector is

$$|00\rangle\langle 00| =: \exp(-B_i^+ b_i) := \vdots \exp(-b_i^+ B_i) \vdots \tag{13}$$

here (and hereafter) the repeated index in a term means summation over 1 to 2, the normal ordering $: \;:$ is for (B_i^+, b_i) and $\vdots \; \vdots$ is for (b_i^+, B_i). Using equation (13) and the IWOP technique, we can prove the following overcompleteness relation:

$$\begin{aligned}\int \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2}{\pi^2} & \exp(-z_i z_i^*) \| z_1, z_2\rangle\langle z_1, z_2| \\ &= \int \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2}{\pi^2} : \exp(-z_i z_i^* + z_i B_i^+ + z_i^* b_i - B_i^+ b_i) := 1\end{aligned} \tag{14}$$

$$\begin{aligned}\int \frac{\mathrm{d}^2 z_1'\, \mathrm{d}^2 z_2'}{\pi^2} & \exp(-z_i' z_i'^*) \| z_1', z_2'\rangle\langle z_1', z_2'| \\ &= \int \frac{\mathrm{d}^2 z_1'\, \mathrm{d}^2 z_2'}{\pi^2} \vdots \exp(-z_i' z_i'^* + z_i' b_i^+ + z_i'^* B_i - b_i^+ B_i) \vdots = 1\end{aligned} \tag{15}$$

where

$$\| z_1, z_2\rangle = \exp(z_i B_i^+)|00\rangle \qquad \langle z_1, z_2| = \langle 00| \exp(z_i^* b_i) \tag{16}$$

$$\| z_1', z_2'\rangle = \exp(z_i' b_i^+)|00\rangle \qquad \langle z_1', z_2'| = \langle 00| \exp(z_i'^* B_i). \tag{17}$$

We now introduce the following states:

$$\| x_1, x_2\rangle = \pi^{-1/2} \exp\left[-\tfrac{1}{2} x_i x_i + \sqrt{2} x_i B_i^+ - \tfrac{1}{2} B_i^+ B_i^+\right]|00\rangle \tag{18}$$

$$\langle x_1, x_2| = \langle 00| \exp\left(-\tfrac{1}{2} x_i x_i + \sqrt{2} x_i b_i - \tfrac{1}{2} b_i b_i\right)\pi^{-1/2} \tag{19}$$

$$\| x_1', x_2'\rangle = \pi^{-1/2} \exp\left[-\tfrac{1}{2} x_i' x_i' + \sqrt{2} x_i' b_i^+ - \tfrac{1}{2} b_i^+ b_i^+\right]|00\rangle \tag{20}$$

$$\langle x_1', x_2'| = \langle 00| \exp\left(-\tfrac{1}{2} x_i' x_i' + \sqrt{2} x_i' B_i - \tfrac{1}{2} B_i B_i\right)\pi^{-1/2}. \tag{21}$$

Letting $\hat{x}_i = (b_i + B_i^+)/\sqrt{2}$ and $\hat{x}_i' = (B_i + b_i^+)/\sqrt{2}$ we have

$$\hat{x}_i \| x_1, x_2\rangle = x_i \| x_1, x_2\rangle \qquad \langle x_1, x_2|\hat{x}_i = \langle x_1, x_2| x_i \tag{22}$$

$$\hat{x}_i' \| x_1', x_2'\rangle = x_i' \| x_1', x_2'\rangle \qquad \langle x_1', x_2'|\hat{x}_i' = \langle x_1', x_2'| x_i'. \tag{23}$$

Performing the following integration by using the IWOP technique we obtain the completeness relations

$$\iint \mathrm{d}x_1\, \mathrm{d}x_2\, \| x_1, x_2\rangle\langle x_1, x_2| = 1 \qquad \iint \mathrm{d}x_1'\, \mathrm{d}x_2'\, \| x_1', x_2'\rangle\langle x_1', x_2'| = 1. \tag{24}$$

Following [18] we derive the f-analogue of two-mode squeeze operators by

$$\begin{aligned}S_2 &= \iint \mathrm{d}x_1\,\mathrm{d}x_2\,\|x_1\cosh\lambda + x_2\sinh\lambda, x_1\sinh\lambda + x_2\cosh\lambda\rangle\langle x_1, x_2| \\ &= \frac{1}{\pi}\iint \mathrm{d}x_1\,\mathrm{d}x_2 : \exp\big[-\cosh^2\lambda(x_1^2+x_2^2) - x_1x_2\sinh 2\lambda \\ &\quad +\sqrt{2}(x_1\cosh\lambda + x_2\sinh\lambda)B_1^+ + \sqrt{2}(x_2\cosh\lambda + x_1\sinh\lambda)B_2^+ \\ &\quad -\tfrac{1}{2}(b_1+B_1^+)^2 - \tfrac{1}{2}(b_2+B_2^+)^2 + \sqrt{2}(x_1b_1+x_2b_2)\big] : \\ &= \exp(B_1^+B_2^+\tanh\lambda)\exp[(B_1^+b_1 + B_2^+b_2 + 1)\ln\operatorname{sech}\lambda]\exp(-b_1b_2\tanh\lambda) \end{aligned} \tag{25}$$

$$\begin{aligned}S_2' &= \iint \mathrm{d}x_1'\,\mathrm{d}x_2'\|x_1'\cosh\lambda + x_2'\sinh\lambda, x_1'\sinh\lambda + x_2'\cosh\lambda\rangle\langle x_1', x_2'| \\ &= \exp(b_1^+b_2^+\tanh\lambda)\exp[(b_1^+B_1 + b_2^+B_2 + 1)\ln\operatorname{sech}\lambda]\exp(-B_1B_2\tanh\lambda).\end{aligned} \tag{26}$$

By means of equations (25) and (26) we can prove that S_2 and S_2' generate the following two-mode squeeze transformations:

$$S_2b_1S_2^{-1} = b_1\cosh\lambda - B_2^+\sinh\lambda \tag{27}$$

$$S_2b_2S_2^{-1} = b_2\cosh\lambda - B_1^+\sinh\lambda \tag{28}$$

$$S_2'B_1S_2'^{-1} = B_1\cosh\lambda - b_2^+\sinh\lambda \tag{29}$$

$$S_2'B_2S_2'^{-1} = B_2\cosh\lambda - b_1^+\sinh\lambda. \tag{30}$$

Operating S_2 and S_2' on the state $|00\rangle$ we obtain two f-analogues of squeezed vacuum states

$$S_2|00\rangle = \operatorname{sech}\lambda\exp(B_1^+B_2^+\tanh\lambda)|00\rangle = \operatorname{sech}\lambda\sum_{n=0}^{\infty}\frac{\tanh^n\lambda}{[f(n)]![f(n)]!}|n,n\rangle \tag{31}$$

$$S_2'|00\rangle = \operatorname{sech}\lambda\exp(b_1^+b_2^+\tanh\lambda)|00\rangle = \operatorname{sech}\lambda\sum_{n=0}^{\infty}\tanh^n\lambda[f(n)]![f(n)]!|n,n\rangle \tag{32}$$

where

$$[f(k)]! = f(1)f(2)\ldots f(k) \qquad [f(0)]! = 1. \tag{33}$$

Operating the displacement operators $D(z_1,z_2) = \exp(z_iB_i^+ - z_i^*b_i)$ and $D(z_1', z_2') = \exp(z_i'b_i^+ - z_i'^*B_i)$ on the squeezed vacuum states $S_2|00\rangle$ and $S_2'|00\rangle$ respectively, we get the f-analogue of coherent squeezed states

$$\begin{aligned}\|z_1,z_2,\lambda\rangle &= D(z_1,z_2)S_2|00\rangle \\ &= \operatorname{sech}\lambda\exp\big[-\tfrac{1}{2}(|z_1|^2+|z_2|^2) + (B_1^+ - z_1^*)(B_2^+ - z_2^*)\tanh\lambda\big]\|z_1,z_2\rangle\end{aligned} \tag{34}$$

$$\begin{aligned}\|z_1',z_2',\lambda\rangle &= D(z_1',z_2')S_2'|00\rangle \\ &= \operatorname{sech}\lambda\exp\big[-\tfrac{1}{2}(|z_1'|^2+|z_2'|^2) + (b_1^+ - z_1'^*)(b_2^+ - z_2'^*)\tanh\lambda\big]\|z_1',z_2'\rangle.\end{aligned} \tag{35}$$

Using the IWOP technique and equation (13) we can prove that the state (34) and the following state:

$$\langle z_1,z_2,\lambda| = \operatorname{sech}\lambda\exp\big[-\tfrac{1}{2}(|z_1|^2+|z_2|^2) + (b_1 - z_1)(b_2 - z_2)\tanh\lambda\big]\langle z_1,z_2| \tag{36}$$

compose the following completeness relation:

$$\int \frac{\mathrm{d}^2 z_1\,\mathrm{d}^2 z_2}{\pi^2} \|z_1, z_2, \lambda\rangle\langle z_1, z_2, \lambda|$$
$$= \int \frac{\mathrm{d}^2 z_1\,\mathrm{d}^2 z_2}{\pi^2} : \exp\big[-|z_1|^2 - |z_2|^2 + z_1 B_1^+ + z_2 B_2^+ + z_1^* b_1 + z_2^* b_2 + (B_1^+ - z_1^*)(B_2^+ - z_2^*)\tanh\lambda + (b_1 - z_1)(b_2 - z_2)\tanh\lambda - B_1^+ b_1 - B_2^+ b_2\big] := 1. \tag{37}$$

Similarly, the state (35) and the state

$$\langle z_1', z_2', \lambda| = \operatorname{sech}\lambda \exp\big[-\tfrac{1}{2}(|z_1'|^2 + |z_2'|^2) + (B_1 - z_1')(B_2 - z_2')\tanh\lambda\big]\langle z_1', z_2'| \tag{38}$$

satisfy the following completeness relation:

$$\int \frac{\mathrm{d}^2 z_1'\,\mathrm{d}^2 z_2'}{\pi^2} \|z_1', z_2', \lambda\rangle\langle z_1', z_2', \lambda| = 1. \tag{39}$$

Generally speaking, there can be any number of f-analogues of squeezed vacuum states and coherent squeezed states corresponding to various choices of the nonlinearity function. In the following we shall confine ourselves to the choice of nonlinearity considered in [8] to describe the motion of a trapped ion. In the present case the nonlinearity function $f(k)$ is given by

$$f(k) = L_k^1(\eta^2)[(k+1)L_k^0(\eta^2)]^{-1} \tag{40}$$

where η is known as the Lamb–Dicke parameter and $L_k^l(x)$ denotes the generalized Lagurre polynomials.

3. Statistical properties of the state $S_2'|00\rangle$

We now study the statistical properties of the state $S_2'|00\rangle$. The second-order correlation function for the ith mode is defined as

$$g_i^{(2)}(0) = \frac{\langle a_i^{+2} a_i^2\rangle}{\langle a_i^+ a_i\rangle^2} \tag{41}$$

$g_i^{(2)}(0) < 1$ means the ith mode exhibits antibunching effects. It is evident that for the state $S_2'|00\rangle$, $g_1^{(2)}(0) = g_2^{(2)}(0)$. The second-order correlation function between two modes

$$g_{12}^{(2)}(0) = \frac{\langle a_1^+ a_1 a_2^+ a_2\rangle}{\langle a_1^+ a_1\rangle\langle a_2^+ a_2\rangle} > 1 \tag{42}$$

shows that the two modes are correlated. If

$$I_0 = \frac{[\langle a_1^{+2} a_1^2\rangle\langle a_2^{+2} a_2^2\rangle]^{1/2}}{|\langle a_1^+ a_1 a_2^+ a_2\rangle|} - 1 < 0 \tag{43}$$

then the Cauchy–Schwartz inequality is violated.

The numerical calculation results for $g_1^{(2)}(0)$ and I_0 are shown in figures 1, 2. From figure 1 we can see that both modes exhibit the antibunching effects, and the antibunching effects are strengthened as the Lamb–Dicke parameter η increases. As the parameter $\lambda > \lambda_0$ (λ_0 depends on the Lamb–Dicke parameter η), the Cauchy–Schwartz inequality is violated (see figure 2). Like the ordinary two-mode squeezed vacuum state, the two modes in the state $S_2'|00\rangle$ are correlated (here omitted).

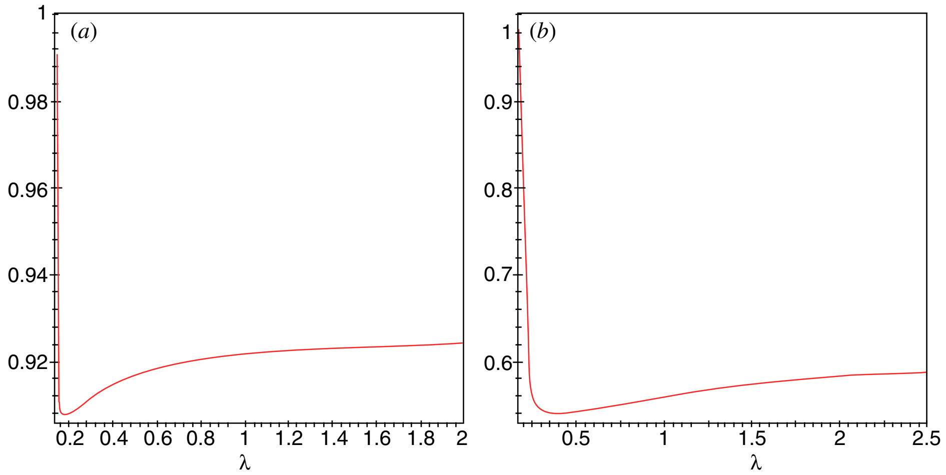

Figure 1. (*a*) $g_1^{(2)}(0)$ versus λ for $\eta = 0.4$. (*b*) $g_1^{(2)}(0)$ versus λ for $\eta = 0.8$.

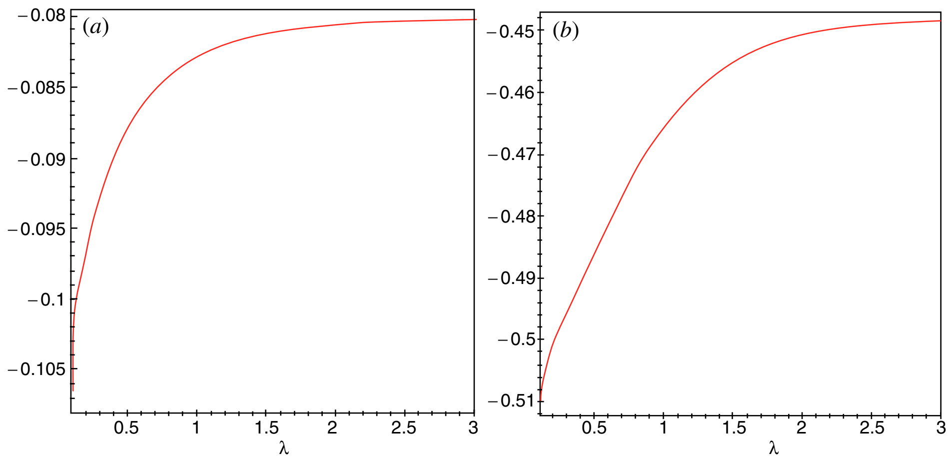

Figure 2. (*a*) I_0 versus λ for $\eta = 0.4$. (*b*) I_0 versus λ for $\eta = 0.8$.

4. Summary and discussion

In this paper, we have generalized the notion of nonlinear coherent states to the two-mode nonlinear squeezed states, which is based on the properties of the inverses of the annihilation operator and the creation operator of f-oscillators, and study their statistical properties. The results show that the introduced two-mode squeezed states have rather different statistical properties from those of the usual squeezed states. These properties depend essentially on the Lamb–Dicke parameter η. In view of their singular properties, states of the type considered might be of great interest, for example in the optical and microwave fields, in molecular vibrations or nuclei vibrations for polyatomic molecules etc. On the other hand, they turned out to be interesting from the point of view of quantum groups too. In fact, the two-mode squeeze operators S_2 and S_2' can be rewritten as $S_2 = \exp(\lambda K_+ - \lambda K_-)$, $S_2' = \exp(\lambda K_+' - \lambda K_-')$, where $K_- = b_1 b_2$, $K_+ = B_1^+ B_2^+$, $K_0 = \frac{1}{2}(N_1 + N_2 + 1)$, and $K_-' = B_1 B_2$, $K_+' = b_1^+ b_2^+$, $K_0 = \frac{1}{2}(N_1 + N_2 + 1)$ are two non-Hermitian two-mode realizations of $SU(1, 1)$ Lie algebra in terms of the f-oscillator.

Using the technique of IWOP of operators we derive some new completeness relations. It should be pointed out that the usual completeness relations are constructed by the bra and ket which are mutually Hermitian conjugate. The completeness relations we obtain here, however, are composed of the bra and ket which are not mutually Hermitian conjugate.

With the recent advances in laser cooling of a trapped ion, it has become possible to realize nonclassical states of the centre-of-mass motion of a single trapped ion. An ion confined in an electromagnetic trap can be regarded as a particle with quantized centre-of-mass motion moving in a harmonic potential. In the Lamb–Dicke limit and the resolved sideband limit, the system can be simplified to a form similar to the Jaynes–Cummings model. As the coupling between the vibrational modes and the external environment is extremely weak, dissipative effects, which are inevitable from cavity damping in the optical regime, can be significantly suppressed for the ion motion. This unique feature thus makes it possible to realize cavity QED experiments without using an optical cavity. Following this approach, nonclassical vibrational states of the trapped ions, such as Fock, coherent squeezed, even and odd coherent, nonlinear coherent, and dark pair coherent states have been proposed. If we consider the quantized motion of a two-level ion that is trapped in a two-dimensional isotropic harmonic potential, the vibrational states of the trapped ions (following the line of thought of de Matos Filho and Vogel [8]) may be related to the states constructed in this paper (this question is under consideration).

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